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ON THE SECOND LOEWY TERM OF PROJECTIVES OF A GROUP ALGEBRA

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ABSTRACT

Let G be a finite group and K a field of prime characteristic p. Let A be an irreducible KG-module and $P_G(A)$ the projective cover of A. In this paper we show that

$$\Phi_p(C_G(A)) \leqq C_G(P_G(A)/P_G(A)J(KG)^2) \leqq C_G(A) \cap \Phi_p(G),$$

where, for a group H, $\Phi_p(H) = \Phi(H \bmod O_{p'}(H))$.

The above formula was already established by Stammbach for A in the principal block of G. Our method, which extends those of [15] and [12], applies indifferently to the general case.

Well-known results on the centralizers in G of the set of the composition factors of $P_G(A)$ or $P_G(A)/P_G(A)J(KG)^2$ are due to Michler [13], Willems [17] and Pahlings [14]. The special case $A = K = \mathbf{F}_p$ is of particular interest and several results on this appear, aside from the already quoted papers, in Aschbacher–Guralnick [1], Brandis [2], Gaschütz [9], Griess–Schmid [4], Gruenberg [6], Kovács [10], Stammbach [16] and Willems [18].

The general results and notation on group theory proceed from [8] and [9], and on cohomological topics from [5] and [7].

Observe that, as it is remarked in [15], the above formula applies also to $\mathrm{Soc}^2(P_G(A))$.

§1. Previous results

We need to apply some results of [12]. As there it is always assumed that $K = A = \mathbf{F}_p$, we begin by extending some of them to the general case.

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1.1. Lemma. *Let V be an irreducible KG-module. Let U be an irreducible $\mathbf{F}_p G$-module such that V is a component of $U^K := U \otimes_{\mathbf{F}_p} K$. Denote V_0 the $\mathbf{F}_p G$-module obtained from V by restriction of the ground-field. Then V_0 is isomorphic to a direct sum of copies of U. In particular $C_G(V) = C_G(U)$.*

Proof. By [9] VII, 1.15 U^K is in particular a completely reducible KG-module, say

$$U^K = V_1 \oplus \cdots \oplus V_r,$$

where $V_1 = V$. Assume that $K = \bigoplus_{i \in I} \mathbf{F}_p$. Then we have

$$\bigoplus_{i \in I} U = (U^K)_0 = (V_1)_0 \oplus \cdots \oplus (V_r)_0.$$

The lemma follows now immediately. □

1.2. Definition. Let V be an irreducible KG-module and let U be an irreducible $\mathbf{F}_p G$-module such that V is a component of U^K. Then we set $D_G(V) := D_G(U)$.

Recall that $D_G(U)$ is defined as follows:

$$D_G(U) = \bigcap \{R \lhd G;\ U \cong_G C_G(U)/R, \text{ which is complemented in } G\},$$

if U is isomorphic to a complemented (chief) factor of G, and

$$D_G(U) := C_G(U)$$

otherwise (see [12]; cf. also [10]).

Then $C_G(U)/D_G(U)$ is the U-crown of G (see [3], [11]).

Definition 1.2 is consistent, as such a U is uniquely determined by V.

We have the following cohomological characterization:

1.3. Proposition. *Let V be an irreducible KG-module. Let N be a normal subgroup of G such that $N \leqq C_G(V)$. Then $N \leqq D_G(V)$ if and only if $H^1(G, V) = H^1(G/N, V)$.*

Proof. Assume that $N \leqq D := D_G(V)$. To conclude $H^1(G, V) = H^1(G/N, V)$ it suffices to show that the inflation $\alpha : H^1(G/D, V) \to H^1(G, V)$ is an epimorphism. Consider the inflation–restriction sequence

$$0 \longrightarrow H^1(G/D, V) \xrightarrow{\alpha} H^1(G, V) \xrightarrow{\beta} \mathrm{Hom}_{G/D}(D, V)$$

relative to the extension of groups $1 \to D \to G \to G/D \to 1$.

We show that $\beta = 0$. Let $\xi \in H^1(G, V)$. Let $V_0 = \bigoplus_{i \in I} U$. Let $\pi_i : V \to U$ the canonical projection corresponding to $i \in I$. We have the commutative and exact diagram:

$$\begin{array}{ccccccc} 0 \longrightarrow & H^1(G/D, V) & \xrightarrow{\alpha} & H^1(G, V) & \xrightarrow{\beta} & \mathrm{Hom}_{G/D}(D, V) \\ & \downarrow (\pi_i)_* & & \downarrow (\pi_i)_* & & \downarrow (\pi_i)_* \\ 0 \longrightarrow & H^1(G/D, U) & \xrightarrow{\alpha'} & H^1(G, U) & \xrightarrow{\beta'} & \mathrm{Hom}_{G/D}(D, U). \end{array}$$

Then $\xi\beta\pi_i = (\xi\beta)(\pi_i)_* = \xi(\pi_i)_*\beta' = 0$, by [12] 2.2. Therefore $\xi\beta = 0$. And $\beta = 0$.

Assume conversely that $H^1(G, V) = H^1(G/N, V)$. We use the canonical inclusion $\pi_i : V \to U$ and follow an analogous way to conclude $H^1(G, U) = H^1(G/N, U)$. [12] 2.2 assures then that $N \leqq D_G(U)$. □

1.4. Definition. Let V be an irreducible KG-module. Let $\xi \in H^1(G, V)$. Let $0 \to V \to E \to K \to 0$ be a short exact sequence of KG-modules, which represents ξ. We set then $C_G(\xi) = C_G(E)$, and:

$$C(G, K) = (V;\ V \text{ is an irreducible } KG\text{-module}, H^1(G, V) \neq 0),$$

$$C_1(G, K) = (V \in C(G, K);\ \exists\, \xi \in H^1(G, V) \text{ with } C_G(\xi) < C_G(V)),$$

$$C_2(G, K) = C(G, K) - C_1(G, K),$$

where we identify isomorphic modules, that is $C(G, K)$, $C_1(G, K)$ and $C_2(G, K)$ consist actually of isomorphism classes of irreducible modules.

(Observe that $C(G, K)$ is the set of the composition factors of $P_G(K)J(KG)/P_G(K)J(KG)^2$. If $K = \mathbf{F}_p$, then we have the sets $C(G)$, $C_1(G)$ — the set of the complemented abelian p-chief factors of G — and $C_2(G)$ of [12].)

1.5. Lemma. *Let V be an irreducible KG-module and U an irreducible $\mathbf{F}_pG$-module such that V is a component of U^K. Then $V \in C(G, K)$ if and only if $U \in C(G, \mathbf{F}_p)$.*

Proof. We must show that $H^1(G, V) \neq 0$ if and only if $H^1(G, U) \neq 0$. As U is a direct summand of V_0, then $H^1(G, U) \neq 0$ implies $H^1(G, V) \neq 0$.

Assume conversely that $H^1(G, U) = 0$. Let d be a derivation from G to V. Let $V_0 = \bigoplus_{i \in I} U$ and $\pi_i : V \to U$ the canonical projection corresponding to $i \in I$. Then $d\pi_i$ is a derivation from G to U. As $H^1(G, U) = 0$, then $d\pi_i$ is an inner derivation. Hence there exists $u_i \in U$ such that $gd\pi_i = u_i g - u_i$ for every $g \in G$. The set $J = (j \in I : gd\pi_j \neq 0 \text{ for some } g \in G)$ is finite, because $\bigoplus_{i \in I} U$ is a sum and G is finite. Therefore $gd = \Sigma_{j \in J}\, gd\pi_j$ for every $g \in G$. So we have

$$gd = \sum_{j\in J} (u_j g - u_j) = \left(\sum_{j\in J} u_j\right) g - \left(\sum_{j\in J} u_j\right).$$

Hence d is an inner derivation, and $H^1(G, V) = 0$. □

1.6. THEOREM. *Let $V \in C(G, K)$ and $U \in C(G, \mathbf{F}_p)$ such that V is a component of U^K. Then $V \in C_1(G, K)$ if and only if U is isomorphic to a complemented chief factor of G.*

PROOF. Assume first that $U \in C_1(G, \mathbf{F}_p)$. Let C/D be the U-crown of G. Let C/R be a chief factor of G such that $D \leqq R$. So U is isomorphic to C/R, which is complemented in G. We have an $\mathbf{F}_p G$-epimorphism $\pi: C/D \to U$ with kernel R/D. Let $\mu: U \to V$ any $\mathbf{F}_p G$-monomorphism. Consider the short exact sequence of groups

$$1 \to C/D \to G/D \to G/C \to 1.$$

The 5 term exact sequence is natural. So we have, by [12] 2.3, the exact and commutative diagram:

$$\begin{array}{ccccccccc}
0 \longrightarrow & H^1(G/C, U) & \longrightarrow & H^1(G, U) & \longrightarrow & \mathrm{Hom}_{G/C}(C/D, U) & \longrightarrow & & 0 \\
& \downarrow \mu_* & & \downarrow \mu_* & & \downarrow \mu_* & & & \downarrow \mu_* \\
0 \longrightarrow & H^1(G/C, V) & \xrightarrow{\alpha} & H^1(G, V) & \xrightarrow{\beta} & \mathrm{Hom}_{G/C}(C/D, V) & \xrightarrow{\Delta} & & H^2(G/C, V).
\end{array}$$

Let now $\phi = \pi\mu_*$. Then $\phi\Delta = 0$. So there exists $\xi \in H^1(G, V)$ such that $\xi\beta = \phi$. By [15] 4.1 we have $C_{G/D}(\xi) = \ker(\xi\beta) = \ker \pi = R/D$. Hence $C_G(\xi) = R$. In particular $V \in C_1(G, K)$.

Assume now that $V \in C_1(G, K)$. Let $\xi \in H^1(G, V)$ such that $C_G(\xi) < C_G(V)$. Consider the above diagram. We have in particular that $\xi\beta \neq 0$. Hence $C \neq D$. Therefore $U \in C_1(G, \mathbf{F}_p)$. □

1.7. COROLLARY. *If $V \in C(G, K)$, then $\bigcap (C_G(\xi); \xi \in H^1(G, V)) = D_G(V)$.*

PROOF. If $V \in C_2(G, K)$ and $\xi \in H^1(G, V)$, then $C_G(\xi) = C_G(V) = D_G(V)$ by 1.5, 1.6, 1.2 and 1.1.

Assume that $V \in C_1(G, K)$ and let $U \in C_1(G, \mathbf{F}_p)$ such that V is a component of U^K. Let C/D be the U-crown of G. Let C/R be a chief factor of G with $D \leqq R$. In the first part of the proof of 1.6 we see that there exists $\xi \in H^1(G, V)$ such that $C_G(\xi) = R$. As C/D is completely reducible we obtain by 1.6 and the Remark (ii) to 4.1 of [15], which we will use in the sequel without explicit mention, that

$$\bigcap(C_G(\xi);\ \xi\in H^1(G, V))\leqq D_G(V).$$

On the other hand, if $\xi\in H^1(G, V)$ we see at the end of the proof of 1.6 that $D_G(V)\leqq C_G(\xi)$. □

We recall the following definitions given in [12]:

$$F_p^a(G)=\bigcap(C;\ C/D \text{ is an abelian } p\text{-crown of } G),$$

$$\Phi_p^a(G)=\bigcap(D;\ C/D \text{ is an abelian } p\text{-crown of } G),$$

$$\theta_p(G)=\bigcap(C_G(U);\ U\in \mathbf{C}_2(G, \mathbf{F}_p)).$$

Then we have:

1.8. Corollary. *One has the following equalities*:

(1) $\bigcap(C_G(V);\ V\in \mathbf{C}_1(G, K))=F_p^a(G)$,

(2) $\bigcap(C_G(\xi);\ \xi\in H^1(G, V),\ V\in \mathbf{C}_1(G, K))=\Phi_p^a(G)$,

(3) $\bigcap(C_G(\xi);\ \xi\in H^1(G, V),\ V\in \mathbf{C}_2(G, K))=\bigcap(C_G(V);\ V\in \mathbf{C}_2(G, K))=\theta_p(G)$,

(4) (Griess–Schmid [4] and Pahlings [14])

$$C_G(P_G(K)J(KG)/P_G(K)J(KG)^2)=F_p(G),$$

(5) (Stammbach [15]) $C_G(P_G(K)/P_G(K)J(KG)^2)=\Phi_p(G)$.

Proof.

(1) $\bigcap(C_G(V);\ V\in \mathbf{C}_1(G, K))=\bigcap(C_G(U);\ U\in \mathbf{C}_1(G, \mathbf{F}_p))$ (by 1.6 and 1.1)
$=F_p^a(G)$. (by 2.7 of [12]).

(2) Is a consequence of the definition of $\Phi_p^a(G)$ and 1.7.

(3) Proceed analogously as in (1).

(4) $C_G(P_G(K)J(KG)/P_G(K)J(KG)^2)=\bigcap(C_G(V);\ V\in \mathbf{C}(G, K))$
$=F_p^a(G)\cap\theta_p(G)=F_p(G)$
(by 2.11 (b) of [12]).

(5) $C_G(P_G(K)/P_G(K)J(KG)^2)=\bigcap(C_G(\xi);\ \xi\in H^1(G, V),\ V\in \mathbf{C}(G, K))$
$=\Phi_p^a(G)\cap\theta_p(G)=\Phi_p(G)$
(by 2.11 (c) of [12]). □

§2. The general case

In this paragraph A is an irreducible KG-module.

Let B be a KG-module, $\xi\in \mathrm{Ext}_{KG}(A, B)$ and $0\to B\to E\to A\to 0$ a short exact sequence of KG-modules, which represents ξ. We set then, by extension of Definition 1.4: $C_G(\xi) = C_G(E)$. And

$$\mathbf{C}(G, A) = (B; B \text{ is an irreducible } KG\text{-module, } \mathrm{Ext}_{KG}(A, B) \neq 0),$$

$$\mathbf{C}_1(G, A) = (B \in \mathbf{C}(G, A);\ \exists \xi \in \mathrm{Ext}_{KG}(A, B) \text{ with } C_G(\xi) < C_G(A) \cap C_G(B)),$$

$$\mathbf{C}_2(G, A) = \mathbf{C}(G, A) - \mathbf{C}_1(G, A).$$

Observe that $\mathbf{C}(G, A)$ is the set of the composition factors of $P_G(A)J(KG)/P_G(A)J(KG)^2$.

In addition we set $N := C_G(A)$.

2.1. Lemma. *Let $B \in \mathbf{C}(G, A)$ and $0 \neq \xi \in \mathrm{Ext}_{KG}(A, B)$. Then one of the following assertions is true*;

(1) $C_G(\xi) = N$.

(2) *The irreducible components of $B\downarrow_N$ belong to $\mathbf{C}(N, K)$.*

Proof. Consider the short exact sequence of groups

$$1 \to N \to G \to G/N \to 1$$

and the inflation–restriction sequence

$$0 \to H^1(G/N, (A^* \otimes_K B)^N) \xrightarrow{\alpha} H^1(G, A^* \otimes_K B) \xrightarrow{\beta} H^1(N, A^* \otimes_K B)^{G/N}.$$

By hypothesis $\xi \in H^1(G, A^* \otimes_K B) = \mathrm{Ext}_{KG}(A, B)$.

If $\xi\beta = 0$, then $0 \neq \xi \in \mathrm{im}\,\alpha$. In particular $B^N \neq 0$. Hence $B^N = B$, as N is a normal subgroup of G and B is an irreducible KG-module. Then $(A^* \otimes_K B)^N = A^* \otimes_K B$. Therefore $\xi \in H^1(G/N, A^* \otimes_K B) = \mathrm{Ext}_{KG/N}(A, B)$. In particular $N \leqq C_G(\xi)$. Hence $N = C_G(\xi)$, because $C_G(\xi) \leqq C_G(A) = N$. And we have (1).

Assume now that $\xi\beta \neq 0$. In particular $H^1(N, A^* \otimes_K B) \neq 0$. As A is a trivial N-module it follows that $H^1(N, B\downarrow_N) \neq 0$. Let W be an irreducible component of $B\downarrow_N$. Then $B\downarrow_N$ is a direct sum of G-conjugates Wg of W, by Clifford's theorem. As $H^1(N, Wg) \cong H^1(N, W)$, it follows that $H^1(N, W) \neq 0$. Hence $W \in \mathbf{C}(N, K)$. And we have (2). □

2.2. Lemma. *Let $B \in \mathbf{C}(G, A)$ and $0 \neq \xi \in \mathrm{Ext}_{KG}(A, B)$. Then one of the following assertions is true*:

(1) $C_G(\xi) = C_N(B)$.

(2) *If W is an irreducible component of $B\downarrow_N$, then $W \in \mathbf{C}_1(N, K)$ and* $\mathrm{core}_G D_N(W) \leqq C_G(\xi) < C_N(B)$.

PROOF. Let $0 \to B \to E \to A \to 0$ be a short exact sequence of KG-modules, which represents ξ. Let $(E_i/B; i \in I)$ and $(B/B_j; j \in J)$ be, respectively, the sets of the irreducible components of $(E/B)\downarrow_N$ and of the irreducible quotients of $B\downarrow_N$.

We first prove that $C_G(E) = \bigcap_{i\in I, j\in J} C_N(E_i/B_j)$. We have that $C_N(E) = N \cap C_G(E) = C_G(E)$, as $C_G(E) \leqq C_G(A) = N$. It is obvious that $C_N(E) \leqq \bigcap_{i\in I, j\in J} C_N(E_i/B_j)$. Now let $g \in \bigcap_{i\in I; j\in J} C_N(E_i/B_j)$. Let $e \in E$. Let $i \in I$ such that $e \in E_i$. Then $eB_j \in E_i/B_j$ for every $j \in J$ (we use multiplicative notation). Then $[g, e] \in B_j$ for every $j \in J$. Therefore $[g, e] = 1$, because $B\downarrow_N$ is completely reducible. Hence $g \in C_N(E)$.

Assume that (1) is not valid. Then there exist $i \in I$ and $j \in J$ such that $C_N(E_i/B_j) < C_N(B/B_j)$. On the contrary we should have

$$C_G(E) = \bigcap_{i\in I, j\in J} C_N(E_i/B_j) = \bigcap_{j\in J} C_N(B/B_j)$$

$$= \bigcap_{g\in G} C_N(B/B_j)^g = C_N(B) \quad \text{(by Clifford's theorem)}$$

contrary to the hypothesis. We thus have a short exact sequence of KN-modules:

$$0 \to W := B/B_j \to E_i/B_j \to E_i/B = K \to 0$$

with $C_N(E_i/B_j) < C_N(W)$. Therefore $W \in C_1(N, K)$. The rest follows immediately. □

2.3. COROLLARY. $\Phi_p^a(C_G(A)) \leqq \bigcap (C_G(\xi); \xi \in \mathrm{Ext}_{KG}(A, B), B \in C_1(G, A))$.

PROOF. $\Phi_p^a(N) = \bigcap (D_N(U); U \in C_1(N, \mathbf{F}_p))$ *is a characteristic subgroup of* N, hence normal in G. □

2.4. LEMMA. *Let $W \in C(N, K)$. Then one of the following assertions is true*:

(1) $W \cong K$.

(2) *There exists $B \in C(G, A)$ such that W is a component of $B\downarrow_N$.*

PROOF. By hypothesis, $H^1(N, W) \neq 0$. By Shapiro's lemma, $H^1(G, W\uparrow^G) \neq 0$. Then there exists a composition factor V of $W\uparrow^G$ such that $H^1(G, V) \neq 0$. $(W\uparrow^G)\downarrow_N$ is a direct sum of G-conjugates of W, by [8] V, 16.10. By the Jordan–Hölder theorem, W is a component of $V\downarrow_N$.

As $H^1(G, V) \neq 0$, then there exists a non-split short exact sequence of KG-modules

(e) $0 \to V \to F \to K \to 0$.

With the diagonal action one obtains the short exact sequence of KG-modules

(e′) $0 \to A \otimes_K V \to A \otimes_K F \to A \to 0$.

$P_G(A)$ is the projective cover of A, hence we have the commutative and exact diagram:

$$\begin{array}{ccccccccc} 0 \to & P_G(A)J(KG) & \to & P_G(A) & \to & A & \to 0 \\ & \downarrow \mu & & \downarrow & & \downarrow = & \\ 0 \to & A \otimes_K V & \to & A \otimes_K F & \to & A & \to 0. \end{array}$$

(1) Assume that $\mu = 0$. Then (e′) splits. Therefore, $A \otimes_K F \cong A \oplus (A \otimes_K V)$. Let $A \downarrow_N \cong \bigoplus_{i \in I} K$. One obtains

$$\bigoplus_{i \in I} F \downarrow_N \cong \left(\bigoplus_{i \in I} K \right) \oplus \left(\bigoplus_{i \in I} V \downarrow_N \right).$$

By the Krull–Schmidt theorem, $F \downarrow_N \cong K \oplus V \downarrow_N$. Consider now the inflation–restriction sequence

$$0 \longrightarrow H^1(G/N, V^N) \xrightarrow{\alpha} H^1(G, V) \xrightarrow{\beta} H^1(N, V)^{G/N}.$$

The sequence (e) corresponds to a non-zero element $\xi \in H^1(G, V)$. As $F \downarrow_N \cong K \oplus V \downarrow_N$, $\xi\beta = 0$. Hence $0 \neq \xi \in \operatorname{im} \alpha$. In particular $V^N \neq 0$. Therefore $N \leqq C_G(V)$, as V is irreducible and N normal in G. Therefore W is the trivial KG-module.

(2) Assume now that $\mu \neq 0$. Let $B = P_G(A)J(KG)/M$ be an irreducible quotient of $P_G(A)J(KG)/\ker \mu$. We obtain a short exact sequence of KG-modules

$$0 \to B \to P_G(A)/M \to A \to 0,$$

which does not split. So $B \in \mathbf{C}(G, A)$.

We have $(A \otimes_K V) \downarrow_N \cong \bigoplus_{i \in I} V \downarrow_N$. W is isomorphic to a component of $B \downarrow_N$, since $\operatorname{im} \mu$ is a submodule of $A \otimes_K V$. □

2.5. Corollary. ([14], Satz 2; [15], 3.1)

$$C_G(A) \cap C_G(P_G(A)J(KG)/P_G(A)J(KG)^2) = F_p(C_G(A)).$$

Proof.

$$C_G(A) \cap C_G(P_G(A)J(KG)/P_G(A)J(KG)^2) = N \cap (\bigcap (C_G(B); B \in \mathbf{C}(G,A))) = \bigcap (C_N(B); B \in \mathbf{C}(G,A)).$$

On the other hand,

$$\bigcap (C_N(W); W \in \mathbf{C}(N,K)) \leqq \bigcap (C_N(B); B \in \mathbf{C}(G,A)) \qquad \text{(by 2.1)}$$
$$\leqq \bigcap (C_N(W); W \in \mathbf{C}(N,K)) \qquad \text{(by 2.4)}.$$

Apply now 1.8(4). □

2.6. Corollary.

$$\bigcap (C_G(\xi); \xi \in \mathrm{Ext}_{KG}(A,B), B \in \mathbf{C}_2(G,A)) = \bigcap (C_G(B); B \in \mathbf{C}_2(G,A)) = \theta_p(C_G(A)).$$

Proof. The first equality follows from the definition of $\mathbf{C}_2(G,A)$.

Let $W \in \mathbf{C}_2(N,K)$. Let U be an irreducible component of W_0. Then $U \in \mathbf{C}_2(N, \mathbf{F}_p)$, by 1.6. In particular $C_N(U) < N$. Hence $C_N(W) < N$, by 1.1. By 2.4 there exists $B \in \mathbf{C}(G,A)$ such that W is a component of $B \downarrow_N$. If $\xi \in \mathrm{Ext}_{KG}(A,B)$, then it follows from 2.2 that $C_G(\xi) = C_N(B)$. Therefore $B \in \mathbf{C}_2(G,A)$. On the other hand, $B \downarrow_N$ is a direct sum of G-conjugates of W, by Clifford's theorem. Hence $C_N(B) = \mathrm{core}_G\, C_N(W)$. Therefore we have:

$$\bigcap (C_G(B); B \in \mathbf{C}_2(G,A)) = \bigcap (\mathrm{core}_G(C_N(W)); W \in \mathbf{C}_2(N,K)) = \theta_p(N),$$

by 1.8. □

2.7. Corollary. $\Phi_p(C_G(A)) \leqq C_G(P_G(A)/P_G(A)J(KG)^2)$.

Proof. It is a consequence of 2.3 and 2.6, because $\Phi_p^a(N) \cap \theta_p(N) = \Phi_p(N)$, by [12] 2.11(c). □

2.8. Lemma. *Let $U \in \mathbf{C}_1(N, \mathbf{F}_p)$ and C/D the U-crown of G. Assume that $D \cap N < C \cap N =: M$. Let M/L be a chief factor of G such that $D \cap N \leqq L$. Then there exist $B \in \mathbf{C}_1(G,A)$ and $\xi \in \mathrm{Ext}_{KG}(A,B)$ such that $C_G(\xi) = L$. Moreover $U \downarrow_N$ and $(B_0) \downarrow_N$ have the same set of irreducible components. If W is such a component, then $W \in \mathbf{C}_1(N,K)$ and*

$$\mathrm{core}_G\, D_N(W) \leqq \bigcap (C_G(\xi); \xi \in \mathrm{Ext}_{KG}(A,B)).$$

Proof. The short exact sequence of groups:

(s) $1 \to U \cong M/L \to G/L \to G/M \to 1$

splits by hypothesis. Let V be an irreducible component of U^K. Let B be an irreducible component of $(A \otimes_K V)/(A \otimes_K V)J(KG)$. As V is irreducible, the canonical projection $A \otimes_K V \to B$ leads to a monomorphism $V \to A^* \otimes_K B$. U is a component of V_0, hence we obtain a G-module monomorphism $\mu: U \to A^* \otimes_K B$. The 5-term exact sequence relative to (s) gives the commutative and exact diagram:

$$\begin{array}{ccccccc} 0 \to H^1(G/M, U) & \to & H^1(G/L, U) & \to & \mathrm{Hom}_{G/M}(U, U) & \xrightarrow{\Delta} & H^2(G/M, V) \\ \downarrow \mu^* & & \downarrow \mu^* & & \downarrow \mu^* & & \downarrow \mu^* \\ 0 \to H^1(G/M, Y) & \to & H^1(G/L, Y) & \xrightarrow{\beta'} & \mathrm{Hom}_{G/M}(U, Y) & \xrightarrow{\Delta'} & H^2(G/M, Y), \end{array}$$

where $Y = A^* \otimes_K B$. (s) splits, hence $1_U\Delta = 0$. Therefore $\mu\Delta' = 0$. Then there exists $\xi \in H^1(G/L, A^* \otimes_K B) = \mathrm{Ext}_{KG/L}(A, B)$ such that $\xi\beta' = \mu$. Therefore $C_{G/L}(\xi) = \ker \xi\beta' = \ker \mu$, by [15] 4.1. Hence $C_G(\xi) = L$. As $M = N \cap C_G(V) = C_G(A \otimes_K V)$, we have $M \leqq C_G(B)$. And we are on the case (2) of Lemma 2.2. □

2.9. Lemma. *Let C/D be an abelian crown of G. Let M be a normal subgroup of G. Then $C \cap M = D \cap M$ if and only if $M \leqq D$.*

Proof. If $C \cap M = D \cap M$, then $C \cap DM = D(C \cap M) = D(D \cap M) = D$, by Dedekind's law. Therefore $CM/DM \cong_G C/D$. Hence

$$M \leqq CM \leqq C_G(CM/DM) = C_G(C/D) = C.$$

Therefore $M = C \cap M = D \cap M$. The converse is obvious. □

2.10. Theorem.

$$\Phi_p(C_G(A)) \leqq C_G(P_G(A)/P_G(A)J(KG)^2) \leqq C_G(A) \cap \Phi_p(G).$$

Proof. Set $X_G(A) = \bigcap (C_G(\xi); \xi \in \mathrm{Ext}_{KG}(A, B),\ B \in \boldsymbol{C}_1(G, A))$. Let $B \in \boldsymbol{C}_1(G, A)$ and $\xi \in \mathrm{Ext}_{KG}(A, B)$. Let W be an irreducible component of $B \downarrow_N$. By 2.1 either $C_G(\xi) = N$ or $W \in \boldsymbol{C}(N, K)$. In this last case W belongs actually to $\boldsymbol{C}_1(N, K)$, by definition of $\boldsymbol{C}_1(G, A)$ and 2.2. And then we have $C_G(\xi) < C_N(B) \leqq C_N(W)$. From 2.4 it follows that $X_G(A) \leqq F_p^a(N)$.

On the other hand 2.8 and 2.9 imply $X_G(A) \leqq \Phi_p^a(G)$. Therefore we have:

$$
\begin{aligned}
&C_G(P_G(A)/P_G(A)J(KG)^2)\\
&= \theta_p(N) \cap X_G(A) && \text{(by definition of } X_G(A) \text{ and 2.6)}\\
&\leqq \theta_p(N) \cap F_p^a(N) \cap \Phi_p^a(G)\\
&= F_p(N) \cap \Phi_p^a(G) && \text{(by [12] 2.11(b))}\\
&= N \cap F_p(G) \cap \Phi_p^a(G)\\
&= N \cap \Phi_p(G) && \text{(again by [12] 2.11(b)).}
\end{aligned}
$$

Finally consider 2.7. □

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